XII.

On the representation of a function by a trigonometric series.

(*Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, vol. 13.)

The following essay on trigonometric series consists of two essentially different parts. The first part contains a history of the research and opinions on arbitrary (graphically given) functions and their representation by trigonometric series. In its composition I was guided by some hints of the famous mathematician, to whom the first fundamental work on this topic was due. In the second part, I examine the representation of a function by a trigonometric series including cases that were previously unresolved. For this, it was necessary to start with a short essay on the concept of a definite integral and the scope of its validity.

History of the question of the representation of an arbitrary function by a trigonometric series.

1.

The trigonometric series named after Fourier, that is, the series of the form

$$\begin{aligned} &a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \\ +\frac{1}{2}\, &b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \end{aligned}$$

play a significant role in those parts of mathematics where arbitrary functions occur. Indeed, there is reason to assert that the most substantial progress in this part of mathematics, that is so important for physics, has depended on a clear insight into the nature of these series. As soon as mathematical research first led to consideration of arbitrary functions, the question arose whether an arbitrary function could be expressed by a series of the above form.

This occurred in the middle of the eighteenth century during the study of vibrating strings, a topic in which the most prominent mathematicians of the time were interested. Their insights about our topic would probably not be represented were it not for the investigation of this problem.

As is well known, under certain hypotheses that conform approximately to reality, the shape of a string under tension that is vibrating in a plane is determined by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

where x is the distance of an arbitrary one of its points from the origin and y is the distance from the rest position at time t. Furthermore α is independent of t, and also of x for a string of uniform thickness.

D'Alembert was the first to give a general solution to this differential equation.

He showed[1] that each function of x and t, which when set in the equation for y yields an identity, must have the form

$$f(x + \alpha t) + \phi(x - \alpha t).$$

This follows by introducing the independent variables $x + \alpha t$, $x - \alpha t$ instead of x and t, whereby

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial^2 y}{\partial t^2} \quad \text{changes into} \quad 4\,\frac{\partial \frac{\partial y}{\partial (x+\alpha t)}}{\partial (x - \alpha t)}.$$

Besides the partial differential equation, which results from the general laws of motion, y must also satisfy the condition that it is always 0 at the endpoints of the string. Thus, if one of these points is at $x = 0$ and the other at $x = \ell$, we have

$$f(\alpha t) = -\phi(-\alpha t), \qquad f(\ell + \alpha t) = -\phi(\ell - \alpha t)$$

and consequently

$$f(z) = -\phi(-z) = -\phi(\ell - (\ell + z)) = f(2\ell + z),$$
$$y = f(\alpha t + x) - f(\alpha t - x).$$

After d'Alembert had succeeded in finding the above for the general solution of the problem, he treated, in a sequel[2] to his paper, the equation

[1] *Mémoires de l'académie de Berlin*, 1747, p. 214.
[2] Ibid. p. 220.

$f(z) = f(2\ell + z)$. That is, he looked for analytic expressions that remained unchanged if z is increased by 2ℓ.

In the next issue of *Mémoires de l'académie de Berlin*[3], Euler made a basic advance, giving a new presentation of d'Alembert's work and recognizing more exactly the nature of the conditions which the function $f(x)$ must satisfy. He noted that, by the nature of the problem, the movement of the string is completely determined, if at some point in time the shape of the string and the velocity are given at each point (that is, y and $\frac{\partial y}{\partial t}$). He showed that if one thinks of the two functions as being determined by arbitrarily drawn curves, then the d'Alembert function $f(z)$ can always be found by a simple geometric construction. In fact, if one assumes that $y = g(x)$ and $\frac{\partial y}{\partial t} = h(x)$ when $t = 0$, then one obtains

$$f(x) - f(-x) = g(x) \quad \text{and} \quad f(x) + f(-x) = \frac{1}{\alpha}\int h(x)\,dx$$

for values of x between 0 and ℓ, and hence obtains the function $f(z)$ between $-\ell$ and ℓ. From this, however, the values of $f(z)$ can be derived for all other values of z using the equation

$$f(z) = f(2\ell + z).$$

This is, represented in abstract but now generally accepted concepts, Euler's determination of the function $f(z)$.

D'Alembert at once protested against this extension of his methods by Euler[4], since it assumed that y could be expressed analytically in t and x.

Before Euler replied to this, Daniel Bernoulli[5] presented a third treatment of this topic, which was quite different from the previous two. Even prior to d'Alembert, Taylor[6] had seen that $y = \sin\frac{n\pi x}{\ell}\cos\frac{n\pi\alpha t}{\ell}$, where n is an integer, satisfies $\frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2}$ and always equals 0 for $x = 0$ and $x = \ell$. From this he explained the physical fact that a string, besides its fundamental tone, can also give the fundamental tone of a string that is $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ as

[3] *Mémoires de l'académie de Berlin*, 1748, p. 69.

[4] *Mémoires de l'académie de Berlin*, 1750, p. 358. 'Indeed, it seems to me, one can only express y analytically in a more general fashion by supposing it is a function of t and x. But with this assumption one only finds a solution of the problem for the case where the different graphs of the vibrating string can be contained in a single equation.'

[5] *Mémoires de l'academie de Berlin*, 1753, p. 147.

[6] Taylor, *De methode incrementorum*.

long (but otherwise similarly constituted). He took his particular solutions as general: he thought that if the pitch of the tone was determined by the integer n, then the vibration of the string would always be as expressed by the equation, or at least very nearly. The observation that a string could simultaneously sound different notes now led Bernoulli to the remark that the string (by the theory) could also vibrate in accordance with the equation

$$y = \sum a_n \sin \frac{n\pi x}{\ell} \cos \frac{n\pi\alpha}{\ell} (t - \beta_n).$$

Further, since all observed modifications of the phenomenon could be explained by this equation, he considered it the most general solution.[7] In order to support this opinion, he examined the vibration of a massless thread under tension, which was weighted at isolated points with finite masses. He showed that the vibrations can be decomposed into a number of vibrations that is always equal to the number of points, each vibration being of the same duration for all masses.

This work of Bernoulli prompted a new paper from Euler, which was printed immediately following it in the *Mémoires de l'académie de Berlin.*[8] He maintained, in opposition to d'Alembert[9], that the function $f(z)$ could be completely arbitrary between $-\ell$ and ℓ. Euler[10] noted that Bernoulli's solution (which he had previously represented as particular) is general if and only if the series

$$\begin{aligned} &a_1 \sin \frac{x\pi}{\ell} + a_2 \sin \frac{2x\pi}{\ell} + \cdots \\ +&\frac{1}{2} b_0 + b_1 \cos \frac{x\pi}{\ell} + b_2 \cos \frac{2x\pi}{\ell} + \cdots \end{aligned}$$

can represent the ordinate of an arbitrary curve for the abcissa x between 0 and ℓ. Now no one doubted at that time that all transformations which could be made with an analytic expression (finite or infinite) would be valid for each value of the variable, or only inapplicable in very special cases. Thus it seemed impossible to represent an algebraic curve, or in general a nonperiodic analytically given curve, by the above expression. Hence Euler thought that the question must be decided against Bernoulli.

[7]Loc. cit., p. 157 section XIII.

[8]*Mémoires de l'académie de Berlin*, 1753, p. 196.

[9]Loc. cit., p. 214

[10]Loc. cit., sections III–X.

The disagreement between Euler and d'Alembert was still unresolved by this. This induced the young, and then little known, mathematician Lagrange to seek the solution of the problem in a completely new way, by which he reached Euler's results. He undertook[11] to determine the vibration of a massless thread which is weighted with an indeterminate finite number of equal masses that are equally spaced. He then examined how the vibrations change when the number of masses grows towards infinity. Although he carried out the first part of this investigation with much dexterity and a great display of analytic ingenuity, the transition from the finite to the infinite left much to be desired. Hence d'Alembert could continue to vindicate the reputation of his solution as the most general by making this point in a note in his *Opuscules Mathématiques*. The opinions of the prominent mathematicians of this time were, and remained, divided on the matter; for in later work everyone essentially retained his own point of view.

In order to finally arrange his views on the problem of arbitrary functions and their representation by trigonometric series, Euler first introduced these functions into analysis, and supported by geometrical considerations, applied infinitesimal analysis to them. Lagrange[12] considered Euler's results (his geometric construction for the course of the vibration) to be correct, but he was not satisfied with Euler's geometric treatment of the functions. D'Alembert,[13] on the other hand, acceded to Euler's way of obtaining the differential equation and restricted himself to disputing the validity of his result, since one could not know for an arbitrary function whether its derivatives were continuous. Concerning Bernoulli's solution, all three agreed not to consider it as general. While d'Alembert,[14] in order to explain Bernoulli's solution as less general than his own, had to assert that an analytically given periodic function cannot always be represented by a trigonometric series, Lagrange[15] believed it possible to prove this.

2.

Almost fifty years had passed without a basic advance having been made in the question of the analytic representation of an arbitrary function. Then

[11] *Miscellanea Taurinensia*, vol. I. Recherches sur la nature et la propagation du son.

[12] *Miscellanea Taurinensia*, vol. II, *Pars math.*, p. 18.

[13] *Opuscules Mathématiques*, d'Alembert. Vol. 1, 1761, p. 16, Sections VII—XX.

[14] *Opuscules Mathématiques*, vol. I, p. 42, Section XXIV.

[15] *Misc. Taur. vol. III, Pars math.*, p. 221, Section XXV.

a remark by Fourier threw a new light on the topic. A new epoch in the development of this part of mathematics began, which soon made itself known in a wonderful expansion of mathematical physics. Fourier noted that in the trigonometric series

$$f(x) = \begin{cases} a_1 \sin x + a_2 \sin 2x + \cdots \\ +\frac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + \cdots , \end{cases}$$

the coefficients can be determined by the formulae

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

He saw that the method can also be applied if the function $f(x)$ is arbitrary. He used a so-called discontinuous function for $f(x)$ (with ordinate a broken line for the abscissa x) and obtained a series which in fact always gives the value of the function.

Fourier, in one of his first papers on heat, which was submitted to the French academy[16] (December 21, 1807) first announced the theorem, that an arbitrary (graphically given) function can be expressed as a trigonometric series. This claim was so unexpected to the aged Lagrange that he opposed it vigorously. There should[17] be another note about this in the archives of the Paris academy. Nevertheless, Poisson refers,[18] whenever he makes use of trigonometric series to represent arbitrary functions, to a place in Lagrange's work on the vibrating string where this method of representation can be found. In order to refute this claim, which can only be explained by the well known rivalry[19] between Fourier and Poisson, we must once again return to Lagrange's treatise, since nothing can be found that is published about these facts by the academy.

In fact, one finds in the place cited[20] by Poisson the formula:

$$`y = 2 \int Y \sin X\pi \, dX \times \sin x\pi + 2 \int Y \sin 2X\pi \, dX \times \sin 2x\pi$$
$$+ 2 \int Y \sin 3X\pi \, dX \times \sin 3x\pi + \text{etc.} + 2 \int Y \sin nX\pi \, dX \times \sin nx\pi,$$

[16] *Bulletin des sciences p. la soc. philomatique*, vol I, p. 112.

[17] From a verbal report of Professor Dirichlet.

[18] Among others, in the expanded *Traité de mécanique* No. 323, p. 638

[19] The review in the *Bulletin des Sciences* on the paper submitted by Fourier to the academy was written by Poisson.

[20] *Misc. Taur.*, vol. III, *Pars math.*, p. 261.

so that when $x = X$, one has $y = Y$, Y being the ordinate corresponding to the abscissa X'.

This formula looks so much like a Fourier series that is easy to confuse them with just a quick glance. However, this appearance arises only because Lagrange uses $\int dX$ where today we would use $\sum \Delta X$. It gives the solution to the problem of determining the finite sine series

$$a_1 \sin x\pi + a_2 \sin 2x\pi + \cdots + a_n \sin nx\pi$$

so that it has given values when x equals

$$\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}.$$

Lagrange denotes the variable by X. If Lagrange had let n become infinitely large in this formula, then certainly he would have obtained Fourier's result. However, if we read through his paper, we see that he was far from believing that an arbitrary function could actually be represented by an infinite sine series. Rather, he had undertaken the whole work because be believed that an arbitrary function could not be expressed by a formula. Concerning trigonometric series, he thought they could be used to represent any analytically given periodic function. Admittedly, it now seems scarcely possible that Lagrange did not obtain Fourier's series from his summation formula. However, this can be explained in that the dispute between Euler and d'Alembert had predisposed him towards a particular opinion about the proper method of proceeding. He thought that the vibration problem, for an indeterminate finite number of masses, must be fully solved before applying limit considerations. This necessitated a rather extensive investigation[21], which was unnecessary if he had been acquainted with the Fourier series.

The nature of the trigonometric series was recognized perfectly correctly by Fourier.[22] Since then these series have been applied many times in mathematical physics to represent arbitrary functions. In each individual case it was easy to convince oneself that the Fourier series really converged to the value of the function. However, it was a long time before this important theorem would be proved in general.

[21] *Misc. Taur.*, vol III, *Pars math.*, p. 251.

[22] *Bulletin d. sc.* vol. I, p. 115. 'The coefficients $a, a', a'', \ldots$, being then determined', etc.

The proof which Cauchy[23] read to the Paris academy on February 27, 1826, is inadequate, as Dirichlet[24] has shown. Cauchy assumed that if x is replaced by the complex argument $x + yi$ in an arbitrary periodic function $f(x)$, then the function is finite for each value of y. However, this only occurs if the function is a constant. It is easy to see that this hypothesis was unnecessary for the later conclusions. It suffices that a function $\phi(x + yi)$ exists which is finite for all positive values of y, whose real part is equal to the given periodic function $f(x)$ when $y = 0$. If one assumes this theorem, which is in fact true,[25] then Cauchy's method certainly leads to the goal; conversely, this theorem can be derived from the Fourier series.

3.

The question of the representation by trigonometric series of everywhere integrable functions with finitely many maxima and minima was first settled rigorously by Dirichlet[26] in a paper of January 1829.

The recognition of the proper way to attack the problem came to him from the insight that infinite series fall into two distinct classes, depending on whether or not they remain convergent when all the terms are made positive. In the first class the terms can be arbitrarily rearranged; in the second, on the other hand, the value is dependent on the ordering of the terms. Indeed, if we denote the positive terms of a series in the second class by

$$a_1, a_2, a_3, \dots,$$

and the negative terms by

$$-b_1,\ -b_2,\ -b_3, \dots,$$

then it is clear that $\sum a$ as well as $\sum b$ must be infinite. For if they were both finite, the series would still be convergent after making all the signs the same. If only one were infinite, then the series would diverge. Clearly now an arbitrarily given value C can be obtained by a suitable reordering of the terms. We take alternately the positive terms of the series until the sum is greater than C, and then the negative terms until the sum is less than C. The deviation from C never amounts to more than the size of the term at

[23] *Mémoires de l'ac. d. sc. de Paris*, vol. VI, p. 603.
[24] *Crelle's Journal für die Mathematik*, vol IV, pp. 157 & 158.
[25] The proof can be found in the inaugural dissertation of the author.
[26] *Crelle's Journal*, vol. IV, p. 157.

the last place the signs were switched. Now, since the numbers a as well as the numbers b become infinitely small with increasing index, so also are the deviations from C. If we proceed sufficiently far in the series, the deviation becomes arbitrarily small, that is, the series converges to C.

The rules for finite sums only apply to the series of the first class. Only these can be considered as the aggregates of their terms; the series of the second class cannot. This circumstance was overlooked by mathematicians of the previous century, most likely, mainly on the grounds that the series which progress by increasing powers of a variable generally (that is, excluding individual values of this variable) belong to the first class.

Clearly the Fourier series do not necessarily belong to the first class. The convergence cannot be derived, as Cauchy futilely attempted,[27] from the rules by which the terms decrease. Rather, it must be shown that the finite series

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin\alpha\, d\alpha\, \sin x + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin 2\alpha\, d\alpha \sin 2x + \cdots$$
$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin n\alpha\, d\alpha\, \sin nx$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos\alpha\, d\alpha \cos x$$
$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos 2\alpha\, d\alpha\, \cos 2x + \cdots + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos n\alpha\, d\alpha \cos nx,$$

or, what is the same, the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, \frac{\sin \frac{2n+1}{2}\,(x-\alpha)}{\sin \frac{x-\alpha}{2}}\, d\alpha,$$

approaches the value $f(x)$ infinitely closely when n increases infinitely.

Dirichlet based this proof on two theorems:

1) If $0 < c \le \pi/2$, then $\int_0^c \phi(\beta)\frac{\sin(2n+1)\beta}{\sin\beta}\, d\beta$ tends to $\frac{\pi}{2}\,\phi(0)$ as n increases to infinity.

2) If $0 < b < c \le \pi/2$, then $\int_b^c \phi(\beta)\frac{\sin(2n+1)\beta}{\sin\beta} d\beta$ tends to 0, as n increases to infinity.

[27] Dirichlet in *Crelle's Journal*, vol IV, p. 158. 'Quoi qu'il en soit de cette première observation, ... à mesure que n croit.'

It is assumed in both cases that the function $\phi(\beta)$ is either always increasing or always decreasing between the limits of integration.

If the function f does not change from increasing to decreasing, or from decreasing to increasing, infinitely often, then using the above theorems the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{\sin\frac{2n+1}{2}(x-\alpha)}{\sin\frac{x-\alpha}{2}}\,d\alpha$$

can clearly be split into a finite number of parts, one of which tends[28] to $\frac{1}{2}f(x+0)$, another to $\frac{1}{2}f(x-0)$, and the others to 0, as n increases to infinity.

It follows from this that a periodic function of period 2π, which

1. is everywhere integrable,

2. does not have infinitely many maxima and minima, and

3. assumes the average of the two one-sided limits when the value changes by a jump,

can be represented by a trigonometric series.

It is clear that a function satisfying the first two properties but not the third cannot be represented by a trigonometric series. A trigonometric series representing such a function, except at the discontinuities, would deviate from it at the discontinuities. Dirichlet's research leaves undecided, whether and when functions can be represented by a trigonometric series that do not satisfy the first two conditions.

Dirichlet's work gave a firm foundation for a large amount of important research in analysis. He succeeded in bringing light to a point where Euler was in error. He settled a question that had occupied many distinguished mathematicians for over 70 years (since 1753). In fact, for all cases of nature, the only cases concerned in that work, it was completely settled. For however great our ignorance about how forces and states of matter vary for infinitely small changes of position and time, surely we may assume that the functions which are not included in Dirichlet's investigations do not occur in nature.

[28]It is easy to prove that the value of a function f, which does not have infinitely many maxima or minima, for increasing or decreasing values of the argument with limit x, either approaches fixed limits $f(x+0)$ and $f(x-0)$ (using Dirichlet's notation in Dove's *Repertorium der Physik*, vol. 1, p. 170); or must become infinitely large.

Nevertheless, there are two reasons why those cases unresolved by Dirichlet seem to be worthy of consideration.

First, as Dirichlet noted at the end of his paper, the topic has a very close connection with the principles of infinitesimal calculus, and can serve to bring greater clarity and rigor to these principles. In this regard the treatment of the topic has an immediate interest.

Secondly, however, the applications of Fourier series are not restricted to research in the physical sciences. They are now also applied with success in an area of pure mathematics, number theory. Here it is precisely the functions whose representation by a trigonometric series was not examined by Dirichlet that seem to be important.

Admittedly Dirichlet promised at the conclusion of his paper to return to these cases later, but that promise still remains unfulfilled. The works by Dirksen and Bessel on the cosine and sine series did not supply this completion. Rather, they take second place to Dirichlet in rigor and generality. Dirksen's paper,[29] (almost simultaneous with Dirichlet's, and clearly written without knowledge of it) was, indeed, in a general way correct. However, in the particulars it contained some imprecisions. Apart from the fact that he found an incorrect result in a special case[30] for the sum of a series, he relied in a secondary consideration on a series expansion[31] that is only possible in particular cases. Hence the proof is only complete for functions whose first derivatives are everywhere finite. Bessel[32] tried to simplify Dirichlet's proof. However, the changes in the proof did not give any essential simplification, but at most clothed it in more familiar concepts, at the expense of rigor and generality.

Hence, until now, the question of the representation of a function by a trigonometric series is only settled under the two hypotheses, that the function is everywhere integrable and does not have infinitely many maxima and minima. If the last hypothesis is not made, then the two integral theorems of Dirichlet are not sufficient for deciding the question. If the first is discarded, however, the Fourier method of determining the coefficients is not applicable. In the following, when we examine the question without any particular assumptions on the nature of the function, the method employed, as we will see, is constrained by these facts. An approach as direct as Dirichlet's is not

[29] *Crelle's Journal*, vol IX, p. 170.
[30] *Loc. cit.*, formula 22.
[31] *Loc. cit.*, section 3.
[32] Schumacher, *Astronomische Nachrichten*, 374 (vol. 16, p. 229.)

possible by the nature of the case.

On the concept of a definite integral and the range of its validity.

4.

Vagueness still prevails in some fundamental points concerning the definite integral. Hence I provide some preliminaries about the concept of a definite integral and the scope of its validity.

Hence first: What is one to understand by $\int_a^b f(x)\,dx$?

In order to establish this, we take a sequence of values $x_1, x_2, \ldots, x_{n-1}$ between a and b arranged in succession, and denote, for brevity, $x_1 - a$ by δ_1, $x_2 - x_1$ by δ_2, $\ldots$, $b - x_{n-1}$ by δ_n, and a positive fraction less than 1 by ϵ. Then the value of the sum

$$S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + \delta_3 f(x_2 + \epsilon_3\delta_3) + \cdots + \delta_n f(x_{n-1} + \epsilon_n\delta_n)$$

depends on the selection of the intervals δ and the numbers ϵ. If this now has the property, that however the δ's and ϵ's are selected, S approaches a fixed limit A when the δ's become infinitely small together, this limiting value is called $\int_a^b f(x)\,dx$.

If we do not have this property, then $\int_a^b f(x)\,dx$ is undefined. In some of these cases, attempts have been made to assign a meaning to the symbol, and among these extensions of the concept of a definite integral there is one recognized by all mathematicians. Namely, if the function $f(x)$ becomes infinitely large when the argument approaches an isolated value c in the interval (a, b), then clearly the sum S, no matter what degree of smallness one may prescribe for δ, can reach an arbitrarily given value. Thus it has no limiting value, and by the above $\int_a^b f(x)\,dx$ would have no meaning. However if

$$\int_a^{c-\alpha_1} f(x)\,dx + \int_{c+\alpha_2}^{b} f(x)\,dx$$

approaches a fixed limit when α_1 and α_2 become infinitely small, then one understands this limit to be $\int_a^b f(x)\,dx$.

Other hypotheses by Cauchy on the concept of the definite integral in the cases where the fundamental concepts do not give a value may be appropriate in individual classes of investigation. These are not generally established, and are hardly suited for general adoption in view of their great arbitrariness.